Journal of Algebra and Its Applications
Vol. 25, No. 6 (2026) 2691001 (5 pages)

DOI: 10.1142/S0219498826910019

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www.worldscientific.com

Corrigendum and Addendum

Finite groups with integer harmonic mean of element orders

[J. Algebra Appl., Vol. 24 (2025), Article ID: 2550083,
DOI: 10.1142/S0219498825500835]

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Received 31 August 2025
Accepted 10 December 2025
Published 30 December 2025

Communicated by Mark L. Lewis

In this short note, we provide some corrections to our previous paper [I. C. Pleşca and M. Tărnăuceanu, Finite groups with integer harmonic mean of element orders, *J. Algebra Appl.* **24** (2025) 2550083]. Moreover, we establish a criterion for the supersolvability of finite groups based on the harmonic mean of element orders.

Keywords: Harmonic mean; element order; supersolvable group.

Mathematics Subject Classification 2020: 20D60, 20D10, 20F16

1. Corrigendum

In [5], we defined the harmonic mean of element orders of a finite group G:

$$h_m(G) = \frac{|G|}{m(G)}, \tag{1}$$

where

$$m(G) = \sum_{a \in G} \frac{1}{o(a)}. \tag{2}$$

Example 1.1. Consider $G = A_4$. Then: $h_m(G) = \frac{72}{31}$.

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Lemma 2.3 of [5] gives a lower bound for the harmonic mean $h_m(G)$ of element orders of a finite group G in terms of $C(G) = \{H \leq G \mid H \text{ cyclic}\}$. The provided limit is

$$\frac{p|G|}{(p-1)|C(G)|+1}, \tag{3}$$

where p is wrongly defined to be the smallest prime divisor of $|G|$. This does not hold. Indeed, if we compute the value of (3) for A_4, we get: $\frac{24}{9} = \frac{72}{27} \geq \frac{72}{31}$.

In what follows, we provide a corrected version of the lemma and new proofs of the results that followed from the initial form of the lemma. We use the same notation as in the original paper.

Lemma 1.2 ([5, Corrected version of Lemma 2.3]). *Let G be a finite group, and p the greatest prime divisor of $|G|$. Then the following inequality holds*:

$$h_m(G) \geq \frac{p|G|}{(p-1)|C(G)|+1}.$$

Equality holds if and only if G is a p-group.

The proof is identical to [5, Lemma 2.3], except for the final step, $\frac{\varphi(d)}{d} \leq \frac{p-1}{p}$, for any order d of a nontrivial element of G. Indeed,

$$\frac{\varphi(d)}{d} = \prod_{q|d, q \text{ prime}} \frac{q-1}{q} = \prod_{q|d, q \text{ prime}} \left(1 - \frac{1}{q}\right) \leq \prod_{q|d, q \text{ prime}} \left(1 - \frac{1}{p}\right) \leq \left(1 - \frac{1}{p}\right).$$

The first inequality holds because the function $t \mapsto 1 - \frac{1}{t}$ is strictly increasing on the positive integers. The second holds because the numbers are smaller than 1.

In [5], Lemma 2.3 has been used to prove Theorems 2.7 and 2.8. They still hold true, with a new proof which is based on the classification of finite groups with many cyclic subgroups given below.

Theorem A ([6, 1]). *Let G be a finite group such that $|C(G)| = |G| - \Delta$ with $\Delta \in \{0, 1, 2, 3\}$.*

(a) *If $\Delta = 0$, then G is an elementary abelian 2-group.*
(b) *If $\Delta = 1$, then G is one of the following groups*: C_3, C_4, S_3, D_8.
(c) *If $\Delta = 2$, then G is one of the following groups*: C_6, $C_2 \times C_4$, D_{12}, $C_2 \times D_8$.
(d) *If $\Delta = 3$, then G is one of the following groups*: C_5, Q_8, D_{10}.

Theorem 2.7 ([5]). *Let G be a finite group. Then $h_m(G) = 2$ if and only if $G \cong C_4$ or $G \cong D_8$.*

Theorem 2.8 ([5]). *The finite non-trivial groups G with $h_m(G) \leq 2$ are $C_2^n, n \in \mathbb{N}, C_3, S_3, C_4$ and D_8.*

Proof of Theorems 2.7 and 2.8. Let G be a finite group with $h_m(G) \leq 2$ and denote by $n_d(G)$ the number of elements of order d in G, $\forall\, d \in \mathbb{N}$. Then

$$\begin{aligned}\frac{|G|}{2} \leq m(G) &= 1 + \frac{n_2(G)}{2} + \frac{n_3(G)}{3} + \cdots \\ &\leq 1 + \frac{n_2(G)}{2} + \frac{n_3(G) + \cdots}{3} \\ &= 1 + \frac{n_2(G)}{2} + \frac{|G| - 1 - n_2(G)}{3}\end{aligned}$$

and so

$$|G| \leq n_2(G) + 4.$$

Since $|C(G)| \geq 1 + n_2(G)$, we get

$$|C(G)| \geq |G| - 3,$$

i.e. G is one of the groups in Theorem A. It is now easy to check that $h_m(G) \leq 2$ if and only if G is an elementary abelian 2-group, C_3, C_4, S_3 or D_8. Also, $h_m(G) = 2$ if and only if G is C_4 or D_8. Thus Theorems 2.7 and 2.8 hold. $\square$

2. Addendum

The method developed in Sec. 1 can be used to prove a criterion for supersolvability of finite groups.

Theorem 2.1. *Let G be a finite group. If $h_m(G) < \frac{72}{31}$, then G is supersolvable. Moreover, if $h_m(G) = \frac{72}{31}$, then either $G \cong A_4$ or G is a generalized dihedral group.*

For the proof of Theorem 2.1, we need the following results.

Theorem B ([7, 3]). *Let G be a non-abelian finite group. If $n_2(G) > \frac{1}{2}|G| - 1$, then one of the following holds*:

(a) *$G \cong D(A)$ is a generalized dihedral group, where A is abelian*;
(b) *$G \cong D_8 \times D_8 \times E$, where E is an elementary abelian 2-group*;
(c) *$G \cong H(r) \times E$, where $H(r)$ is a central product of r copies of D_8 and E is an elementary abelian 2-group*;
(d) *$G \cong S(r) \times E$, where $S(r) = \langle x_1, y_1, ..., x_r, y_r, z \mid x_i^2 = y_i^2 = z^2 = 1$, all pairs of generators commute except $[z, x_i] = x_i y_i\rangle$ and E is an elementary abelian 2-group.*

In particular, a finite group G with $n_2(G) > \frac{1}{2}|G| - 1$ is supersolvable.

Theorem C ([2, 4]). *Every finite group of order $n = \prod_{i=1}^{t} p_i^{\alpha_i}$ $(p_1 < p_2 < \cdots < p_t)$ is supersolvable if and only if*:

(1) *For all $1 \leq i \leq t$, the distinct prime factors of $\gcd(n, \psi(p_i^{\alpha_i}))$ are the same as those of $\gcd(n, p_i - 1)$.*

(2) *If there exists $i \neq k$ such that $p_i \leq \alpha_k$, then the following hold*:

(a) *There does not exist a prime p_j such that $p_i|(p_j - 1)$ and $p_j|(p_k - 1)$, and*
(b) $\alpha_i \leq 2$, *and if* $\alpha_i = 2$ *then* $p_i^2|(p_k - 1)$.

Corollary 2.2. *From the theorem above, it follows immediately that p-groups (where p is prime) or groups with the order free of squares are supersolvable. The latter is also observed in* [2].

Proof of Theorem 2.1. Assume that $h_m(G) < \frac{72}{31}$. Then, similarly with the proof in Sec. 1, we have

$$\begin{aligned}\frac{31}{72}|G| < m(G) &= 1 + \frac{n_2(G)}{2} + \frac{n_3(G)}{3} + \cdots \\ &\leq 1 + \frac{n_2(G)}{2} + \frac{n_3(G) + \cdots}{3} \\ &= 1 + \frac{n_2(G)}{2} + \frac{|G| - 1 - n_2(G)}{3}\end{aligned}$$

implying that

$$n_2(G) > \frac{7}{12}|G| - 4.$$

If $|G| \geq 36$, then $\frac{7}{12}|G| - 4 \geq \frac{1}{2}|G| - 1$ and so $n_2(G) > \frac{1}{2}|G| - 1$. Then G is supersolvable by Theorem B.

Assume that $|G| \leq 35$ and G is not supersolvable. From Corollary 2.2, the possible values for $|G|$ are $12, 18, 20, 24$. Applying the complete result from Theorem C, we get $|G| \in \{12, 24\}$. Since A_4 is the unique nonsupersolvable group of order 12, if $|G| = 12$, we get $G \cong A_4$. Thus $h_m(G) = \frac{72}{31}$, a contradiction. Similarly, if $|G| = 24$ we get $G \cong S_4$, $G \cong A_4 \times C_2$ or $G \cong \mathrm{SL}(2,3)$ and so $h_m(G) = \frac{72}{29} > \frac{72}{31}$, $h_m(G) = \frac{48}{17} > \frac{72}{31}$ or $h_m(G) = \frac{24}{7} > \frac{72}{31}$, a contradiction. Thus G is supersolvable in all cases.

Assume now that $h_m(G) = \frac{72}{31}$. Then

$$2 \text{ divides } |G| \quad \text{and} \quad 3 \text{ divides } |G|. \tag{4}$$

By repeating the above reasoning, we obtain

$$n_2(G) \geq \frac{7}{12}|G| - 4$$

and therefore $n_2(G) > \frac{1}{2}|G| - 1$ for $|G| \geq 37$. In this case, G is a generalized dihedral group by Theorem B. If $|G| \leq 36$, then by (4), $|G| \in \{6, 12, 18, 24, 30, 36\}$. By computing the harmonic mean of element orders for all groups of these orders using the software GAP [8], we conclude that the unique possibility is $G \cong A_4$. This completes the proof. □

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